Mathematical Induction

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1. Introduction

The principle of mathematical induction has been used for about 350 years. It was familiar to Fermat, in a disguised form, and the first clear statement seems to have been made by Pascal in proving results about the arrangement of numbers now known as Pascal's Triangle. There are many applications of inductive arguments and the aim of my talk is to give some examples, illustrating why this method has become an indispensable tool for mathematicians.

We begin with a general form of the principle. Let $p_1, p_2, p_3, \ldots$ be statements or propositions, each of which may be true or false.

Principle :

Suppose that (i) p_1 *is true and that, for* $n \geq 1$, (ii) $p_n \Longrightarrow p_{n+1}$, *then* $p_1, p_2, p_3, \ldots$ *are all true.*

Perhaps the most familiar applications are concerned with proving statements like the following.

Example 1 : $\quad p_n \quad : \quad 1 + 2 + \ldots + n = \frac{1}{2}n(n+1).$

Proof. $\quad p_1 \quad : \quad 1 = \frac{1}{2}.1.2$ is true.

Now assume that p_n is true for some $n \geq 1$. Then we have

$$\begin{aligned} 1 + 2 + \ldots + n + (n+1) &= \frac{1}{2}n(n+1) + (n+1) \\ &= \frac{1}{2}(n+1)(n+2). \end{aligned}$$

Lecture given to the Society on 9 March 1988.

In other words,

$$p_n \Longrightarrow p_{n+1}.$$

A slightly harder exercise is to prove that

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1). \tag{1}$$

2. Historical Background

The historical material in this section is based on the book by Boyer [2]. It is remarkable that Fermat published hardly anything on the theory of numbers, but he made some very penetrating notes in the margin of his copy of a 1621 edition of the *Arithmetica* of Diophantus. Some of his theorems were proved by a method that he called ***infinite descent*** and he used it with great ingenuity. However, we can illustrate the method quite easily by proving a classical result.

Example 2: $\sqrt{2}$ is irrational.

Proof. We start with an assumption that

$$\sqrt{2} = \frac{k_1}{k_2},$$

where k_1 and k_2 are positive integers. This will lead to a contradiction which shows that there is no such ratio.

The assumption means that

$$k_1^2 = 2k_2^2,$$

so k_1 is even and $k_1 > k_2$. Now write $k_1 = 2k_3$, so that $k_2^2 = 2k_3^2$ and we have

$$\sqrt{2} = \frac{k_2}{k_3}.$$

By repeating the argument, we can obtain an equation

$$\sqrt{2} = \frac{k_n}{k_{n+1}},$$

for every $n \geq 1$, where $k_1, k_2, \ldots$ are positive integers and $k_1 > k_2 > \ldots$. This infinite descent gives the required contradiction.

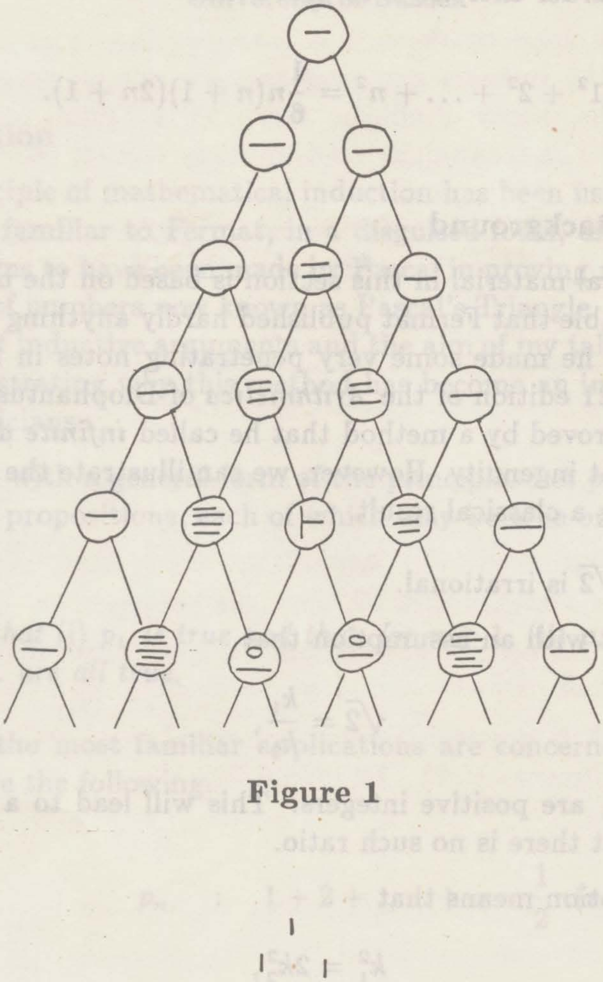

Figure 1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

. . . .

Figure 2

Figures 1 and 2 give two different sketches of what is misleadingly called Pascal's Triangle. The first is a Chinese version copied from a diagram that appeared in the *Ssu-yüan yü-chien* (*Precious Mirror of the Four Elements*) by Chu Shih-chieh in 1303. Chu disclaims credit for the triangle and it seems likely that it originated in China about 1100. Note the use of rod numerals and the zero symbol in Figure 1: see the recent Presidential Address [3] by Lam Lay Yong. It is worth remarking that formulae for the summation of series, such as (1), also appeared without proof in the *Precious Mirror*.

Of course, both figures represent the same mathematical object. The reason that the triangle is associated with Pascal is that, in 1654, he gave a clear explanation of the method of induction and used it to prove some new results about the triangle. In fact, the construction of this infinite triangle is recursive so, with hindsight, the method now seems very natural. Let the symbol $\binom{n}{r}$ represent the rth number in the n-th row of the triangle, $r = 0, 1, 2, \ldots, n$. It is convenient to give the label 0 to the first row, so that

$$\binom{0}{0} = \binom{1}{0} = \binom{1}{1} = 1.$$

Then the triangle is constructed by using the relation

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}. \qquad (2)$$

In other words, each entry is obtained by adding together the pair of numbers immediately above it in the previous row. We are using the relation (2) to define the symbols $\binom{n}{r}$ here. However, it is easy to recover the usual formula.

Example 3 : $$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof. In this case, let us take p_n in the Principle of Induction to include all the above equations associated with the integer n, for $r = 0, 1, 2, \ldots, n$. We have already noted that p_0 and p_1 are both true, so now consider a component of the proposition p_{n+1}, assuming that p_n is true. By using (2) and p_n, we obtain

$$\begin{aligned}\binom{n+1}{r+1} &= \binom{n}{r} + \binom{n}{r+1} \\ &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!} \\ &= \frac{n!}{(r+1)!(n-r)!}(r+1+n-r) \\ &= \frac{(n+1)!}{(r+1)!(n+1-(r+1))!}.\end{aligned}$$

Similarly, it is a straightforward matter to verify that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n. \tag{3}$$

My final remark on the historical background of the method is to note that the Principle was included in 1889 as one of Peano's axioms for the natural numbers, thereby recognising it as one of the foundations of arithmetic.

3. A Diversion.

Inductive arguments are not always straightforward and the following anecdote contains one that is plausible, but false. I came across it in a book of mathematical puzzles, where it was presented without suggesting that there was anything wrong.

Example 4: The Executioner's Tale.

Many years ago, one Friday, in court, a prisoner was convicted of a crime and sentenced to death. The executioner visited him in his cell and offered some hope of freedom. "As it happens", he said, "I am allowed some discretion in my work and I rather enjoy a gamble, occasionally. In your case, the execution is scheduled for next week and I have written the day : Monday, Tuesday, . . . , or Saturday, on a paper sealed in this envelope. I will visit you here early on Monday and then on the following days, if necessary, and ask whether you know the day of your execution. If you answer correctly at your first attempt, then you can go free but, otherwise, I must do my job".

The following Monday when the executioner arrived to ask his question, the prisioner replied immediately, "Yes, it must be today." "What makes you say that ?" said the executioner and this was the prisoner's argument. "Consider the situation on Saturday morning. If you arrive then, I shall be certain that is the appointed day, so it must be earlier. Now consider Friday. If you ask me then, I will be sure of the answer because we have eliminated Saturday. Having excluded the last two days, we can repeat the argument for Thursday, and so on. By proceeding backwards in time, we can eliminate the days until we are left with Monday as the only possibility". The prisoner seemed well pleased with his conclusion and, to be fair, the executioner did not betray any emotion as he handed over the envelope to be opened - it was Wednesday !

There are several confusing features in the above argument, more than enough to invalidate the conclusion. However, the idea of using induction backwards over time is a very useful one and this is the subject of our final illustration.

4. Shortest Paths in a Network

As a research student, I became involved in the application of inductive techniques to optimization problems, developing the ideas expounded in Bellman's book [1]. There are now many applications involving a wide range of mathematical models. Some indication of the growth in this field can be found in more recent books by Whittle [4].

Unfortunately, there is no time to do justice to modern developments and a single example will have to be enough. Consider the problem of finding a path between two vertices in a network so that the distance along the path is a minimum. Suppose the network consists of vertices 1, 2, ..., t, where t is the target to be reached from vertex 1. Some, but not all pairs of vertices are directly linked by an edge and the distance d_{ij} between i and j is given for these pairs. For simplicity, let us assume that there is always a path between any two vertices; its length is obtained by adding the d_{ij} over the corresponding sequence of edges. For any vertex i, the number of paths from i to t is finite. Let f_i be the length of the shortest path. Thus, $f_t = 0$ and it is easy to see that, for $i < t$,

$$f_i = \min_j \left\{ d_{ij} + f_j \right\}. \tag{4}$$

The minimisation on the right of this equation is equivalent to choosing a direction of departure from i : the index j runs over all vertices directly linked to i by an edge. Suppose that all the given distances d_{ij} are positive. Then it can be shown that the system of equations (4) has a unique solution for the shortest lengths f_1, f_2, ..., f_{t-1}, with $f_t = 0$. If we can determine this solution, then it is easy to find a suitable path from 1 to t by following directions that attain the minimum in (4) at every vertex encountered on the way. The solution is constructed by backwards induction : more precisely, the values of the f_i are determined in increasing order. This is best demonstrated by looking at a particular case.

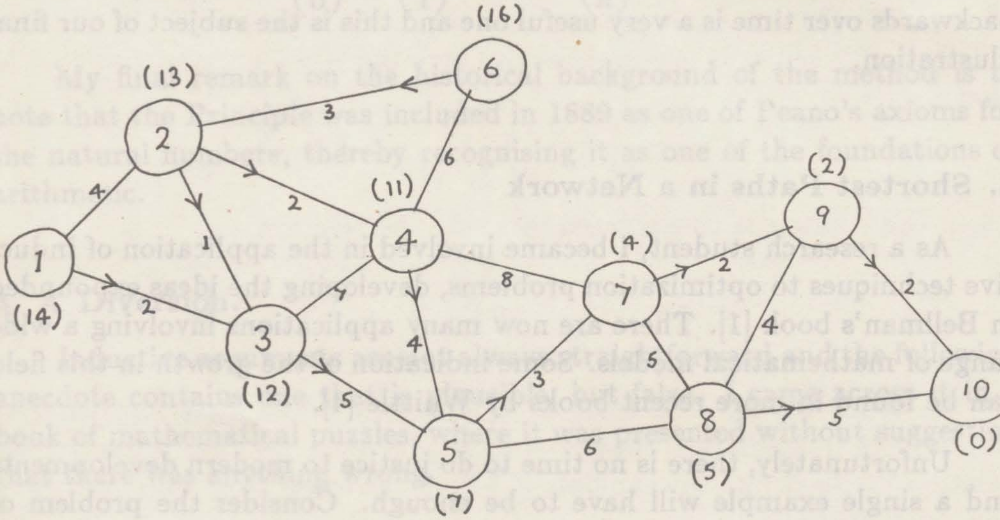

Figure 3

Example 5: Vertices $i = 1, 2, \ldots, 10$ are shown in Figure 3 and each edge is marked with the distance d_{ij}. The length of the shortest path to $t = 10$ is shown as (f_j) near the initial vertex. Note that the solution is constructed by working from right to left on the diagram :

$i =$	10	9	7	8	5	4	3	2	1	6
$f_i =$	0	2	4	5	7	11	12	13	14	16

Finally, we note that the required path from 1 to 10 is determined by following the arrows :

$$1 \longrightarrow 3 \longrightarrow 5 \longrightarrow 4 \longrightarrow 9 \longrightarrow 10$$

and this attains the value

$$f_1 = 2 + 5 + 3 + 2 + 2 = 14.$$

References

[1] Bellman, R., *Dynamic programming,* Princeton University Press, 1957.

[2] Boyer, C. B., *A history of mathematics,* Wiley, New York, 1968.

[3] Lam Lay Yong, The Chinese rod numeral legacy and its impact on mathematics, *Mathematical Medley,* 15 (2), 1987, 52-59.

[4] Whittle, P., *Optimisation over time,* Vol.1, 1982, Vol.2, 1983, Wiley.